Det Kgl. Danske Videnskabernes Selskab.
Mathematisk-fysiske Meddelelser. **XIX**, 9.

UNDOR REPRESENTATION OF THE FIVE-DIMENSIONAL MESON THEORY

BY

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I KOMMISSION HOS EJNAR MUNKSGAARD
1942

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Printed in Denmark.
Bianco Lunos Bogtrykkeri A/S.

In 1935, Yukawa[1] pointed out that short range nuclear forces may be accounted for by a field of a new kind corresponding to the existence of particles—now generally denoted as mesons—with a rest mass intermediary between those of the electron and the proton, and with an integral spin. Since then, the meson field theory has been treated by a great number of authors and has undergone an important development in various respects.

Originally, Yukawa used a scalar field function implying the mesons to have spin 0. Such a formalism gives, however, a repulsion between the nucleons in a ^3S state of the deuteron in contradiction to the experiments which show that the ground state of the deuteron is just a ^3S state. The difficulty was overcome in the further development of the theory when, according to a formalism due to Proca[2], the field was described by means of a vector function. This form of the theory which leads to the value 1 of the spin of the mesons was developed by Yukawa and Sakata[3], Fröhlich, Heitler and Kemmer[4], Bhabha[5], and Stückelberg[6]. Assuming that the spin of the meson does not exceed 1, Kemmer[7] was able to show that the meson field can be described by four—and only by four—different wave-functions characterized by their covariance properties. Besides the mentioned "scalar" and "vector" functions, "pseudoscalar" and "pseudovector" wave-functions or arbitrary combinations of all of them are possible.

In his first paper, Yukawa considered charged mesons, only, but in the subsequent development it has been necessary also to assume the existence of neutral mesons in the theory, since it was shown by collision experiments that the interaction—at nuclear distances—between two protons is approximately the same as that between a proton and a neutron[8]. As found by Kemmer[9], this experimentally observed charge independence of the nuclear forces can be accounted for in combining the wave-functions of the charged and the neutral mesons in a certain symmetrical scheme in which the nucleons interact equally strongly with neutral, positive and negative mesons.

In a paper by Møller and Rosenfeld[10] (in the following quoted as MR.) it was shown, however, that, when introducing the charged and neutral mesons in the mentioned symmetrical way, it is necessary to combine the "vector" theory with the "pseudoscalar" theory in order to avoid the appearance of singular terms in the static interaction between nucleons and, thus, to obtain a basis for an unambiguous description of the stationary states of atomic nuclei.

The charge independence of the nuclear forces may, of course, also be explained by using neutral meson fields, only. A theory on these lines has been put forward by Bethe[11] who used a vector field function for the description of the neutral mesons. In order to get finite results in the calculations following from such a theory it is, however, necessary to "cut off" at small distances in a more or less arbitrary way. To obtain an unambiguous theory for the energy levels of the deuteron without having recourse to such a procedure it is, again, necessary to combine two types of meson fields, viz. a "scalar" and a "pseudovector" field (cf. MR. p. 34, footnote). This theory which, in the following, will be called the "neutral meson theory" can in certain respects be regarded as dual to the "symmetrical meson theory" treated in MR.

The symmetrical theory has been brought into an equivalent and particularly compact form by BELINFANTE [12] who has written the 15 field equations as one single "undor" equation showing a close analogy to the DIRAÇ equation of the electron. The "undors" introduced by BELINFANTE are quantities which under LORENTZ transformations behave like products of DIRAC wave-functions (four-spinors).

Irrespective of the form in which the symmetrical theory is written, the field equations contain four universal constants, f'_1, f'_2, g'_1 and g'_2, mutually independent apart from the condition

$$(f'_2)^2 = (g'_2)^2$$

which is unavoidable if the singular terms shall cancel each other in the nuclear interaction. The circumstance that the two kinds of fields appearing in the theory are in no direct connection with each other and, furthermore, the occurrence of a large number of independent constants constitute an unsatisfactory feature of the theory. Recently, however, MØLLER has shown in a paper [13] (quoted in the following as M.) that the field equations of the vector and the pseudoscalar theory can be comprised in one set of five-dimensional field equations which become invariant with respect to five-dimensional rotations on condition that the independence of the constants is restricted by the relations

$$f'_1 = g'_1 \text{ and } f'_2 = -g'_2.$$

The advantages of the representation used by BELINFANTE as well as the simplification obtained by a five-dimensional treatment of the theory make it desirable to generalize the undor description of the meson theory to the case of five dimensions, a generalization which will be performed in the present paper. In this way, we obtain a rather concise and clear form of the theory and, at the same time, the relation between the neutral and the symmetrical theory is elucidated from a new point of view.

The first sections of this paper are devoted to a more detailed account of some of the earlier investigations mentioned above. Thus, section 1 contains the fundamental equations for the symmetrical theory in five dimensions, in section 2, a certain representation of the Dirac matrices used in the following, the so-called Kramers representation, is introduced and, in section 3, a short account of the general transformation formulae for infinitesimal and finite rotations in five dimensions is given.

Next, section 4 contains the definition and the simplest properties of the four-dimensional undors introduced by Belinfante as well as of their five-dimensional generalization with the alterations which are made necessary by the condition of invariance with respect to five-dimensional rotations. Finally, the undors are expressed by the tensor field quantities and an example is given in order to illustrate the way in which calculations with undors are running.

After these preparations, it is possible to set up, in section 5, the undor field equation of the symmetrical theory and, in section 6, the undor field equation of the neutral meson theory which only by a change of sign differs essentially from the former equation. In section 7, the "adjoint" undor is introduced and expressed in terms of the original undor elements. Furthermore, the corresponding differential equation for this adjoint undor is derived. Now, in section 8, it is shown that the field equation for the symmetrical theory may be derived from a variational principle, the corresponding Lagrangeian assuming a rather elegant form in the undor formulation. The wave-equations of the nucleons and the expression for the current vector are derived.

This formalism is especially well suited to be used for the description of the symmetrical theory but, as shown in section 9, not for the description of the neutral theory. Thus, the impossibility of obtaining in such a way a Lagrangeian principle for the neutral theory in five dimensions would seem to offer an argument in preference of the symmetrical theory.

1. Fundamental equations of the five-dimensional symmetrical meson theory.

In the five-dimensional meson theory the 10 field equations of the vector theory together with the 5 equations of the pseudoscalar theory are comprised in the following equations (cf. M. (12)):

$$\left.\begin{aligned} \boldsymbol{G}_{\mu\nu} &= D_\mu \boldsymbol{U}_\nu - D_\nu \boldsymbol{U}_\mu + \boldsymbol{S}_{\mu\nu} && \text{(a)} \\ D_\nu \boldsymbol{G}_{\mu\nu} + \kappa^2 \boldsymbol{U}_\mu &= \boldsymbol{M}_\mu && \text{(b)} \\ \mu, \nu &= 0, 1, 2, 3, 4; \quad \kappa = \frac{M_m c}{\hbar} \\ D_\mu &= \frac{\partial}{\partial x_\mu} \end{aligned}\right\} \quad (1)$$

where $\boldsymbol{U}_\mu$ and $\boldsymbol{M}_\mu$ are five-vectors, $\boldsymbol{G}_{\mu\nu}$ and $\boldsymbol{S}_{\mu\nu}$ are antisymmetrical five-tensors in the $(x_0, x_1, x_2, x_3, x_4)$-space, and M_m is the mass of the meson. A heavy printed letter indicates that the quantities in question have three independent components in isotopic space so that, for instance, $\boldsymbol{G}_{\mu\nu}$ signifies the symbolic vector:

$$\boldsymbol{G}_{\mu\nu} = (G_{\mu\nu\mathbf{1}},\ G_{\mu\nu\mathbf{2}},\ G_{\mu\nu\mathbf{3}}).$$

We have, thus, to do with three non-interfering meson fields, the index **3** referring to the neutral field, while the indices **1** and **2** refer to the real and imaginary part of the complex field representing the charged mesons. The symbols $\boldsymbol{G}$ and $\boldsymbol{U}$ describe the field quantities, while $\boldsymbol{S}$ and $\boldsymbol{M}$ denote the densities of the source distribution defined by

$$\left.\begin{aligned} \boldsymbol{M}_{\mu} &= g_1\, \psi^{\dagger}\, \boldsymbol{\tau}\, \gamma_{\mu}\, \psi \\ \boldsymbol{S}_{\mu\nu} &= \frac{g_2}{2\,\varkappa}\, \psi^{\dagger}\, \boldsymbol{\tau}\, [\gamma_{\mu}, \gamma_{\nu}]\, \psi \end{aligned}\right\} \tag{2}$$

where $\boldsymbol{\tau}$ is the isotopic spin vector, ψ the wave-function of the nucleons, and its adjoint $\psi^{\dagger}$ is given by

$$\psi^{\dagger} = i\psi^{*}\beta. \tag{3}$$

Further, $[\gamma_{\mu}, \gamma_{\nu}]$ is the commutator of the quantities γ_{μ} used in M. and explicitely written down in equations (5) of the next section.

The equations (1) are evidently invariant with respect to the group of five-dimensional rotations. This group includes, of course, the complete Lorentz group of transformations in the $(x_1,\ x_2,\ x_3,\ x_4)$-space under which the field quantities of the vector theory and of the pseudoscalar theory are transformed separately. For a general five-dimensional rotation, however, the field quantities of these two theories are mixed up.

2. The Kramers representation.

Let ρ_1, ρ_2, ρ_3, σ_1, σ_2, σ_3 be the ordinary Dirac matrices, where ρ_3 and σ_3 are diagonal ("Dirac representation").

Introducing now the variables ρ_x, ρ_y, ρ_z, σ_x, σ_y, σ_z by the following relations

$$\left.\begin{aligned} \rho_x &= \rho_3 & \sigma_x &= \sigma_1 \\ \rho_y &= -\rho_2 & \sigma_y &= \sigma_2 \\ \rho_z &= \rho_1 & \sigma_z &= \sigma_3 \end{aligned}\right\} \quad (4)$$

it is seen that the new variables satisfy the same algebraic relations as the original variables, e. g.

$$\rho_x \rho_y = i\rho_z.$$

In the so-called KRAMERS representation which we, following BELINFANTE, shall use in the present paper, ρ_z and σ_z are chosen to be diagonal so that the matrices of ρ_x, ρ_y, ρ_z, σ_x, σ_y, σ_z in the KRAMERS representation are the same as the matrices of ρ_1, ρ_2, ρ_3, σ_1, σ_2, σ_3 in the DIRAC representation.

For reference, we write down in the KRAMERS representation some of the quantities which will be used in the following:

$$\left.\begin{aligned} \vec{\alpha} &= \rho_z \vec{\sigma} \\ \beta &= \rho_x \\ \vec{\gamma} &= -i\beta\vec{\alpha} = -\rho_y \vec{\sigma} \\ \gamma_4 &= \beta = \rho_x \\ \gamma_0 &= \gamma_1\gamma_2\gamma_3\gamma_4 = -\rho_z \end{aligned}\right\} \quad (5)$$

3. Rotations in the five-dimensional space.

For an infinitesimal rotation in the five-dimensional space,

$$\left. x'_\mu = x_\mu + \sum_{\nu=0}^{4} \varepsilon_{\mu\nu} x_\nu ; \qquad \varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu} \right\} \quad (6)$$

a four-spinor transforms according to the formula (cf. M. (9), (9'))

$$\psi' = S\psi \tag{7}$$

where

$$S = 1 + \frac{1}{2}\sum_{\mu,\nu=0}^{4} q_{\mu\nu}\,\varepsilon_{\mu\nu} \tag{8}$$

$$q_{\mu\nu} = \frac{1}{4}\left[\gamma_\mu, \gamma_\nu\right] \tag{9}$$

or, in the Kramers representation,

$$\{q_{\mu\nu}\} = \begin{Bmatrix} 0 & -\frac{i}{2}\rho_x\sigma_x & -\frac{i}{2}\rho_x\sigma_y & -\frac{i}{2}\rho_x\sigma_z & -\frac{i}{2}\rho_y \\ \frac{i}{2}\rho_x\sigma_x & 0 & \frac{i}{2}\sigma_z & -\frac{i}{2}\sigma_y & \frac{i}{2}\rho_z\sigma_x \\ \frac{i}{2}\rho_x\sigma_y & -\frac{i}{2}\sigma_z & 0 & \frac{i}{2}\sigma_x & \frac{i}{2}\rho_z\sigma_y \\ \frac{i}{2}\rho_x\sigma_z & \frac{i}{2}\sigma_y & -\frac{i}{2}\sigma_x & 0 & \frac{i}{2}\rho_z\sigma_z \\ \frac{i}{2}\rho_y & -\frac{i}{2}\rho_z\sigma_x & -\frac{i}{2}\rho_z\sigma_y & -\frac{i}{2}\rho_z\sigma_z & 0 \end{Bmatrix} \tag{10}$$

The transformation equations for a finite rotation in the (x_α, x_β)-plane are obtained by substituting (8) into

$$S(\varphi + d\varphi) = S(\varphi)\cdot S(d\varphi).$$

We get thus the equation

$$S(\varphi + d\varphi) = (1 + q_{\alpha\beta}\,d\varphi)\,S(\varphi)$$

with the only continuous solution

$$S = e^{q_{\alpha\beta}\varphi} \tag{11}$$

to be used in (7).

4. Undors.

Undors are, according to the definition of BELINFANTE, quantities transforming as products of four-component DIRAC wave-functions (four-spinors). The number of four-spinor factors in the corresponding transformation scheme is called the rank of the undor. An undor of the first rank is thus simply a four-spinor.

In the following, we shall deal with undors of the second rank, i. e. quantities with 16 components $\Psi_{k_1 k_2}$ transforming as products $\psi_{k_1} \psi_{k_2}$ (k_1, $k_2 = 1, 2, 3, 4$). It will be convenient to arrange the 16 components into a square scheme

$$\Psi = \left| \begin{array}{cccc} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ \Psi_{21} & \Psi_{22} & \Psi_{23} & \Psi_{24} \\ \Psi_{31} & \Psi_{32} & \Psi_{33} & \Psi_{34} \\ \Psi_{41} & \Psi_{42} & \Psi_{43} & \Psi_{44} \end{array} \right| \tag{12}$$

which, however, is not to be confounded with a matrix representing an operator.

In our considerations we shall only need operators which act either on the first or on the second index of the undor. We characterize such operators by providing their matrices with the suffix $^{(1)}$ or $^{(2)}$, respectively. Operators with the matrices $A^{(1)}$ and $A^{(2)}$, for instance, will transform the undor Ψ according to the formulae

$$\left. \begin{aligned} (A^{(1)}\Psi)_{k_1 k_2} &= \sum_{k'=1}^{4} (k_1 \,|\, A \,|\, k') \, \Psi_{k' k_2} \\ (A^{(2)}\Psi)_{k_1 k_2} &= \sum_{k'=1}^{4} (k_2 \,|\, A \,|\, k') \, \Psi_{k_1 k'} . \end{aligned} \right\} \tag{13}$$

In accordance with (7), the transformation of an undor of the second rank by a rotation in the five-dimensional space is given by

$$\Psi' = S^{(1)} S^{(2)} \Psi. \tag{14}$$

It can be shown that, if we only consider the subgroup of Lorentz transformations in the (x_1, x_2, x_3, x_4)-space, the transformation (14) will have the special property that the elements inside one of the four subsquares $(\Psi_{11}, \Psi_{12}, \Psi_{21}, \Psi_{22})$, $(\Psi_{31}, \Psi_{32}, \Psi_{41}, \Psi_{42})$, $(\Psi_{13}, \Psi_{14}, \Psi_{23}, \Psi_{24})$ and $(\Psi_{33}, \Psi_{34}, \Psi_{43}, \Psi_{44})$ are not mixed up with elements of the other squares. This is just the case treated by Belinfante in his four-dimensional theory. He was able to show that an undor of the second rank comprises—with respect to the complete Lorentz group—a scalar, a four-vector, an antisymmetrical tensor of the second rank, a pseudo-four-vector, and a pseudo-scalar. Furthermore, he deduced a possible correspondance between the 16 components of these quantities and the 16 undor elements. Now, the meson fields and the source densities are described just by the tensors mentioned above, a fact which makes the undors so useful in the treatment of the meson theory. In the correspondance chosen by Belinfante, a symmetrical undor contains only the quantities appearing in the vector meson theory (the Proca field), while an antisymmetrical undor contains the quantities of the pseudoscalar theory, so that a general undor will just represent the combination of fields dealt with in MR. This correspondance is, however, not unique since another possible correspondance would be obtained by changing the sign of all elements in one or more of the four subsquares in (12).

Passing to the five-dimensional theory, the field variables of the vector and the pseudoscalar theories are, as mentioned in section 1, comprised to a five-vector and an antisymmetrical five-tensor of the second rank (cf. M. (10), (10′), (11), (11′)). Now, if we do not confine ourselves to LORENTZ transformations but consider the full group of five-dimensional rotations, we find that the transformation (14) mixes up elements from different subsquares of the undor scheme (12). This means that some of the arbitrariness in the mentioned correspondance between undor elements and field quantities is removed.

It is thus impossible to make a straightforward generalization of the correspondance set up by BELINFANTE in replacing the field quantities of the four-dimensional theory by the corresponding quantities in five dimensions. A simple calculation shows that this would lead to obvious contradictions. Consequently, we have to find a new correspondance between the 16 undor elements, on one hand, and the antisymmetrical tensor $\boldsymbol{G}_{\mu\nu}$, and the five-vector $\boldsymbol{U}_{\mu}$, on the other hand. Since the last mentioned quantities together have 15 components only, we add a scalar $\boldsymbol{K}$ in order to obtain the complete representation of the 16 independent undor elements. The criterion of the consistance of such a correspondance is that the transformation (14) transforms the quantities $\boldsymbol{G}_{\mu\nu}$, $\boldsymbol{U}_{\mu}$ and $\boldsymbol{K}$ as the components of a tensor, a vector and a scalar, respectively.

A straightforward calculation shows that such a correspondance can be obtained from the correspondance scheme set up by BELINFANTE by a direct generalization to five dimensions followed by a change of sign in the two subsquares forming the last two columns of the undor scheme. This correspondance is given below, in equation (15), for the meson fields and, in equation (16), for the sources giving rise to these fields.

$$\boldsymbol{\Psi}_{k_1 k_2} = \left| \begin{matrix} \boldsymbol{G}_{23}+\boldsymbol{G}_{14}+i\boldsymbol{G}_{13}-i\boldsymbol{G}_{24} & -\boldsymbol{G}_{12}-\boldsymbol{G}_{34}+i\kappa\boldsymbol{U}_0-i\boldsymbol{K} & -\boldsymbol{G}_{01}+i\boldsymbol{G}_{02}-\kappa\boldsymbol{U}_1+i\kappa\boldsymbol{U}_2 & \boldsymbol{G}_{03}-i\boldsymbol{G}_{04}+\kappa\boldsymbol{U}_3-i\kappa\boldsymbol{U}_4 \\ -\boldsymbol{G}_{12}-\boldsymbol{G}_{34}-i\kappa\boldsymbol{U}_0+i\boldsymbol{K} & -\boldsymbol{G}_{23}-\boldsymbol{G}_{14}+i\boldsymbol{G}_{13}-i\boldsymbol{G}_{24} & \boldsymbol{G}_{03}+i\boldsymbol{G}_{04}+\kappa\boldsymbol{U}_3+i\kappa\boldsymbol{U}_4 & \boldsymbol{G}_{01}+i\boldsymbol{G}_{02}+\kappa\boldsymbol{U}_1+i\kappa\boldsymbol{U}_2 \\ -\boldsymbol{G}_{01}+i\boldsymbol{G}_{02}+\kappa\boldsymbol{U}_1-i\kappa\boldsymbol{U}_2 & \boldsymbol{G}_{03}+i\boldsymbol{G}_{04}-\kappa\boldsymbol{U}_3-i\kappa\boldsymbol{U}_4 & \boldsymbol{G}_{23}-\boldsymbol{G}_{14}+i\boldsymbol{G}_{13}+i\boldsymbol{G}_{24} & -\boldsymbol{G}_{12}+\boldsymbol{G}_{34}-i\kappa\boldsymbol{U}_0-i\boldsymbol{K} \\ \boldsymbol{G}_{03}-i\boldsymbol{G}_{04}-\kappa\boldsymbol{U}_3+i\kappa\boldsymbol{U}_4 & \boldsymbol{G}_{01}+i\boldsymbol{G}_{02}-\kappa\boldsymbol{U}_1-i\kappa\boldsymbol{U}_2 & -\boldsymbol{G}_{12}+\boldsymbol{G}_{34}+i\kappa\boldsymbol{U}_0+i\boldsymbol{K} & -\boldsymbol{G}_{23}+\boldsymbol{G}_{14}+i\boldsymbol{G}_{13}+i\boldsymbol{G}_{24} \end{matrix} \right| \tag{15}$$

$$\boldsymbol{X}_{k_1 k_2} = \left| \begin{matrix} -\kappa\boldsymbol{S}_{23}-\kappa\boldsymbol{S}_{14}-i\kappa\boldsymbol{S}_{13}+i\kappa\boldsymbol{S}_{24} & \kappa\boldsymbol{S}_{12}+\kappa\boldsymbol{S}_{34}-i\boldsymbol{M}_0+i\kappa\boldsymbol{C} & \kappa\boldsymbol{S}_{01}-i\kappa\boldsymbol{S}_{02}+\boldsymbol{M}_1-i\boldsymbol{M}_2 & -\kappa\boldsymbol{S}_{03}+i\kappa\boldsymbol{S}_{04}-\boldsymbol{M}_3+i\boldsymbol{M}_4 \\ \kappa\boldsymbol{S}_{12}+\kappa\boldsymbol{S}_{34}+i\boldsymbol{M}_0-i\kappa\boldsymbol{C} & \kappa\boldsymbol{S}_{23}+\kappa\boldsymbol{S}_{14}-i\kappa\boldsymbol{S}_{13}+i\kappa\boldsymbol{S}_{24} & -\kappa\boldsymbol{S}_{03}-i\kappa\boldsymbol{S}_{04}-\boldsymbol{M}_3-i\boldsymbol{M}_4 & -\kappa\boldsymbol{S}_{01}-i\kappa\boldsymbol{S}_{02}-\boldsymbol{M}_1-i\boldsymbol{M}_2 \\ \kappa\boldsymbol{S}_{01}-i\kappa\boldsymbol{S}_{02}-\boldsymbol{M}_1+i\boldsymbol{M}_2 & -\kappa\boldsymbol{S}_{03}-i\kappa\boldsymbol{S}_{04}+\boldsymbol{M}_3+i\boldsymbol{M}_4 & -\kappa\boldsymbol{S}_{23}+\kappa\boldsymbol{S}_{14}-i\kappa\boldsymbol{S}_{13}-i\kappa\boldsymbol{S}_{24} & \kappa\boldsymbol{S}_{12}-\kappa\boldsymbol{S}_{34}+i\boldsymbol{M}_0+i\kappa\boldsymbol{C} \\ -\kappa\boldsymbol{S}_{03}+ik\boldsymbol{S}_{04}+\boldsymbol{M}_3-i\boldsymbol{M}_4 & -\kappa\boldsymbol{S}_{01}-ik\boldsymbol{S}_{02}+\boldsymbol{M}_1+i\boldsymbol{M}_2 & \kappa\boldsymbol{S}_{12}-\kappa\boldsymbol{S}_{34}-i\boldsymbol{M}_0-i\kappa\boldsymbol{C} & \kappa\boldsymbol{S}_{23}-\kappa\boldsymbol{S}_{14}-i\kappa\boldsymbol{S}_{13}-i\kappa\boldsymbol{S}_{24} \end{matrix} \right| \tag{16}$$

In the last formula, $\boldsymbol{C}$ denotes a scalar which must be included in the scheme similarly to the scalar $\boldsymbol{K}$ in (15).

From the equations (2), (3), (5) and the equation

$$\boldsymbol{C} = \frac{g_0}{\kappa} \psi^\dagger \tau \psi, \tag{17}$$

which is the only possible expression for a scalar source, it is seen that

$$f_{op}^{-1} \boldsymbol{X}_{k_1 k_2} = 2 (\psi^\dagger \sigma_y)_{k_1} \tau \psi_{k_2} \tag{18}$$

where f_{op}^{-1} is an operator dividing $\boldsymbol{C}$, $\boldsymbol{M}_\mu$, and $\boldsymbol{S}_{\mu\nu}$ by g_0, g_1 and g_2, respectively.

A simple calculation shows that the correspondance (15) satisfies the criterion mentioned above. As an example, we shall bring the calculation for a rotation in the (x_0, x_1)-plane. The corresponding operator S in equation (14) is, according to (10) and (11),

$$S = \cos \frac{\varphi}{2} - i \rho_x \sigma_x \sin \frac{\varphi}{2}.$$

From (14) we then get

$$\boldsymbol{\Psi}'_{11} = \boldsymbol{\Psi}_{11} \cos^2 \frac{\varphi}{2} - \boldsymbol{\Psi}_{44} \sin^2 \frac{\varphi}{2} - i (\boldsymbol{\Psi}_{14} + \boldsymbol{\Psi}_{41}) \sin \frac{\varphi}{2} \cos \frac{\varphi}{2}$$

$$\boldsymbol{\Psi}'_{12} = \boldsymbol{\Psi}_{12} \cos^2 \frac{\varphi}{2} - \boldsymbol{\Psi}_{43} \sin^2 \frac{\varphi}{2} - i (\boldsymbol{\Psi}_{13} + \boldsymbol{\Psi}_{42}) \sin \frac{\varphi}{2} \cos \frac{\varphi}{2}$$

... etc. (16 eqq.)

or, from (15),

$$\boldsymbol{G}'_{23} + \boldsymbol{G}'_{14} + i \boldsymbol{G}'_{13} - i \boldsymbol{G}'_{24} =$$
$$= \boldsymbol{G}_{23} + \boldsymbol{G}_{14} \cos \varphi + i \boldsymbol{G}_{13} \cos \varphi - i \boldsymbol{G}_{24} - i \boldsymbol{G}_{03} \sin \varphi - \boldsymbol{G}_{04} \sin \varphi$$
$$- \boldsymbol{G}'_{12} - \boldsymbol{G}'_{34} + i \kappa \boldsymbol{U}'_0 - i \boldsymbol{K}' =$$
$$= - \boldsymbol{G}_{12} \cos \varphi - \boldsymbol{G}_{34} + i \kappa \boldsymbol{U}_0 \cos \varphi - i \boldsymbol{K} + \boldsymbol{G}_{02} \sin \varphi + i \kappa \boldsymbol{U}_1 \sin \varphi$$

... etc. (16 eqq.).

Solving these equations we get

$$\begin{aligned} \boldsymbol{G}'_{23} &= \boldsymbol{G}_{23} \\ \boldsymbol{G}'_{14} &= \boldsymbol{G}_{14}\cos\varphi - \boldsymbol{G}_{04}\sin\varphi \\ \boldsymbol{G}'_{13} &= \boldsymbol{G}_{13}\cos\varphi - \boldsymbol{G}_{03}\sin\varphi \\ \boldsymbol{G}'_{24} &= \boldsymbol{G}_{24} \\ \boldsymbol{G}'_{12} &= \boldsymbol{G}_{12}\cos\varphi - \boldsymbol{G}_{02}\sin\varphi \\ \boldsymbol{G}'_{34} &= \boldsymbol{G}_{34} \\ &\ldots \text{etc.} \end{aligned} \qquad \begin{aligned} \boldsymbol{U}'_0 &= \boldsymbol{U}_0\cos\varphi + \boldsymbol{U}_1\sin\varphi \\ &\ldots \text{etc.} \\ \boldsymbol{K}' &= \boldsymbol{K} \end{aligned}$$

i. e. the well-known transformation formulae for a tensor, a vector and a scalar under the rotation in question.

5. The undor field equation of the symmetrical field theory.

Having obtained a correspondance between the tensor quantities of the meson theory and the elements of the undors $\boldsymbol{\Psi}$ and $\boldsymbol{X}$, we may now comprise the 15 equations (1) and a trivial scalar equation in one undor equation closely analogous to the Dirac equation of the electron:

$$2\kappa\boldsymbol{\Psi} + 2\boldsymbol{X} + \left(\gamma_\mu^{(1)} - \gamma_\mu^{(2)}\right) D_\mu \boldsymbol{\Psi} = 0 \tag{19}$$

which, of course, corresponds to 16 equations between the undor elements.

In fact, since the operator $\gamma_\mu D_\mu$ has the representation

$$\gamma_\mu D_\mu = \begin{Bmatrix} -D_0 & 0 & iD_3 + D_4 & iD_1 + D_2 \\ 0 & -D_0 & iD_1 - D_2 & -iD_3 + D_4 \\ -iD_3 + D_4 & -iD_1 - D_2 & D_0 & 0 \\ -iD_1 + D_2 & iD_3 + D_4 & 0 & D_0 \end{Bmatrix} \tag{20}$$

the introduction of (15), (16) and (20) into (19) gives the equations

$$\begin{aligned}\kappa(\boldsymbol{G}_{23}-\boldsymbol{S}_{23})+\kappa(\boldsymbol{G}_{14}-\boldsymbol{S}_{14})+i\kappa(\boldsymbol{G}_{13}-\boldsymbol{S}_{13})-\\ -i\kappa(\boldsymbol{G}_{24}-\boldsymbol{S}_{24})+\kappa(iD_3+D_4)(\boldsymbol{U}_1-i\boldsymbol{U}_2)+\\ +\kappa(iD_1+D_2)(-\boldsymbol{U}_3+i\boldsymbol{U}_4)=0\end{aligned}$$

$$(k_1=1,\ k_2=1)$$

$$\begin{aligned}-\kappa(\boldsymbol{G}_{12}-\boldsymbol{S}_{12})-\kappa(\boldsymbol{G}_{34}-\boldsymbol{S}_{34})+i(\kappa^2\boldsymbol{U}_0-\boldsymbol{M}_0)-i\kappa(\boldsymbol{K}-\boldsymbol{C})+\\ +iD_3(\boldsymbol{G}_{03}-i\kappa\boldsymbol{U}_4)+D_4(i\boldsymbol{G}_{04}-\kappa\boldsymbol{U}_3)+\\ +iD_1(\boldsymbol{G}_{01}-i\kappa\boldsymbol{U}_2)+D_2(i\boldsymbol{G}_{02}-\kappa\boldsymbol{U}_1)=0\end{aligned}$$

. . . etc. (16 eqq.) $(k_1=1,\ k_2=2)$

with the solution

$$\begin{aligned}\boldsymbol{G}_{23}-\boldsymbol{S}_{23}&=D_2\boldsymbol{U}_3-D_3\boldsymbol{U}_2\\ \boldsymbol{G}_{14}-\boldsymbol{S}_{14}&=D_1\boldsymbol{U}_4-D_4\boldsymbol{U}_1\\ \boldsymbol{G}_{13}-\boldsymbol{S}_{13}&=D_1\boldsymbol{U}_3-D_3\boldsymbol{U}_1\\ \boldsymbol{G}_{24}-\boldsymbol{S}_{24}&=D_2\boldsymbol{U}_4-D_4\boldsymbol{U}_2\\ \boldsymbol{G}_{12}-\boldsymbol{S}_{12}&=D_1\boldsymbol{U}_2-D_2\boldsymbol{U}_1\\ \boldsymbol{G}_{34}-\boldsymbol{S}_{34}&=D_3\boldsymbol{U}_4-D_4\boldsymbol{U}_3\end{aligned}$$

. . . etc.

$$\begin{aligned}\kappa^2\boldsymbol{U}_0+D_1\boldsymbol{G}_{01}+D_2\boldsymbol{G}_{02}+\\ +D_3\boldsymbol{G}_{03}+D_4\boldsymbol{G}_{04}=\boldsymbol{M}_0\end{aligned}$$

. . . etc.

$$\boldsymbol{K}=\boldsymbol{C}$$

i. e. the equations (1) and the trivial scalar equation

$$\boldsymbol{K}=\boldsymbol{C}.$$

The equation (19) differs from the undor equation of Belinfante (formula (20), p. 26) by the sign of $\gamma_\mu^{(2)}$ in accordance with the change of sign in the correspondance equations (15) and (16). On account of the well-known properties of the Dirac matrices γ_μ, the equation (19) is invariant with respect to five-dimensional rotations.

6. The undor field equation of the neutral meson theory.

Like the equation (19), the undor equation

$$2\kappa\Psi+2X+\left(\gamma_\mu^{(1)}+\gamma_\mu^{(2)}\right)D_\mu\Psi=0 \qquad (21)$$

is invariant with respect to rotations in five-dimensional space. As it will be shown in the following, (21) represents the field equations of the neutral meson theory. Consequently, the undor quantities Ψ and X are not printed in heavy letters in this section, since the neutral theory only deals with one single component in the isotopic spin space (cf. Section 1).

Substituting (15) and (16) in (21) we obtain the 16 invariant field equations

$$\left.\begin{aligned} &K - C + D_\mu U_\mu = 0 && \text{(a)}\\ &M_\mu = \varkappa^2 U_\mu + D_\mu K && \text{(b)}\\ &\varkappa(G_{\mu\nu} - S_{\mu\nu}) - (-1)^{I[\mu,\nu,\iota,\varkappa,\lambda]}(D_\iota G_{\varkappa\lambda} + D_\lambda G_{\iota\varkappa} + D_\varkappa G_{\lambda\iota}) = 0 && \text{(c)} \end{aligned}\right\} \quad (22)$$

where $I[\mu,\nu,\iota,\varkappa,\lambda]$ is the number of inversions in the permutation μ, ν, ι, $\varkappa$, λ of the numbers 0, 1, 2, 3, 4.

The equations (22) may also be written as

$$\left.\begin{aligned} &K - C + \mathrm{Div}\, U = 0 && \text{(a)}\\ &M = \varkappa^2 U + \mathrm{Grad}\, K && \text{(b)}\\ &\varkappa(G - S) = \mathrm{Rot}\, G && \text{(c)} \end{aligned}\right\} \quad (23)$$

where Div and Grad are the straightforward generalizations of these differential operators to five dimensions, and Rot is defined by

$$D_\iota G_{\varkappa\lambda} + D_\lambda G_{\iota\varkappa} + D_\varkappa G_{\lambda\iota} = \mathrm{pot}_{\iota\varkappa\lambda}\, G = (-1)^{I[\iota,\varkappa,\lambda,\mu,\nu]}\, \mathrm{Rot}_{\mu\nu}\, G. \quad (24)$$

If we express the antisymmetrical tensor of the second rank $G_{\mu\nu}$ by an antisymmetrical tensor $G_{\iota\varkappa\lambda}$ of the third rank by means of a formula analogous to (24) we get, instead of (23 c), the tensor equation of the third rank

$$\varkappa(G - S) = \mathrm{pot}\, G. \quad (23\,\mathrm{d})$$

Supposing now that the variables in (23) are independent of the coordinate x_0, we find that these equations reduce to the four-dimensional equations

$$\left.\begin{aligned} & K - C + D_k U_k = 0 && \text{(a)} \\ & M_k = \kappa^2 U_k + D_k K && \text{(b)} \\ & M_0 = \kappa^2 U_0 && \text{(c)} \\ & \kappa(G_{klm} - S_{klm}) = D_k G_{lm} + D_l G_{mk} + D_m G_{kl} && \text{(d)} \\ & \kappa(G_{kl} - S_{kl}) = D_r G_{klr}, && \text{(e)} \end{aligned}\right\} \quad (25)$$

Latin indices running only from 1 to 4. The equations (25 d) and (25 e) are obtained from (23 d) for ι, κ, $\lambda \neq 0$ and $\iota = 0$, respectively. The equations (25) are just the field equations of the neutral meson theory in four dimensions.

We have, thus, for the scalar-pseudovector theory in five dimensions in (21) obtained an undor representation similar to that given by (19) for the vector-pseudoscalar theory.

7. The adjoint undor.

We define the adjoint $\Psi^\dagger$ to an undor Ψ of the second rank in a similar way as it was done in the case of an undor of the first rank by the equation (3):

$$\Psi^\dagger = -\Psi^* \beta^{(1)} \beta^{(2)}. \qquad (26)$$

Consequently, $\Psi^\dagger$ transforms under a transformation (6) according to the formula

$$\Psi^{\dagger\prime} = \Psi^\dagger (S^{(1)})^{-1} (S^{(2)})^{-1} \qquad (27)$$

so that $\Psi^\dagger \Psi$ is an invariant. As usual, multiplication from the right by operators $A^{(1)}$ and $A^{(2)}$ is defined by

$$\left.\begin{aligned}\left(\Phi A^{(1)}\right)_{k_1 k_2} &= \sum_{k'=1}^{4} \Phi_{k' k_2} \left(k' \mid A \mid k_1\right) \\ \left(\Phi A^{(2)}\right)_{k_1 k_2} &= \sum_{k'=1}^{4} \Phi_{k_1 k'} \left(k' \mid A \mid k_2\right)\end{aligned}\right\} \tag{28}$$

whence

$$(\Phi A) \Psi = \Phi (A \Psi). \tag{29}$$

Remembering that the field quantities $\boldsymbol{G}_{kl}$ and $\boldsymbol{U}_k$ $(k, l = 0, 1, 2, 3)$ are real, while $\boldsymbol{G}_{k4}$, $\boldsymbol{U}_4$ and $\boldsymbol{K}$ are purely imaginary, we get from (15) by a simple calculation that the elements of $\boldsymbol{\Psi}^{\dagger}_{k_1 k_2}$ can in a simple way be expressed by the elements of $\boldsymbol{\Psi}_{k_1 k_2}$, viz.:

$$\boldsymbol{\Psi}^{\dagger} = \left|\begin{array}{cccc} \boldsymbol{\Psi}_{22} & -\boldsymbol{\Psi}_{12} & \boldsymbol{\Psi}_{42} & -\boldsymbol{\Psi}_{32} \\ -\boldsymbol{\Psi}_{21} & \boldsymbol{\Psi}_{11} & -\boldsymbol{\Psi}_{41} & \boldsymbol{\Psi}_{31} \\ \boldsymbol{\Psi}_{24} & -\boldsymbol{\Psi}_{14} & \boldsymbol{\Psi}_{44} & -\boldsymbol{\Psi}_{34} \\ -\boldsymbol{\Psi}_{23} & \boldsymbol{\Psi}_{13} & -\boldsymbol{\Psi}_{43} & \boldsymbol{\Psi}_{33} \end{array}\right| \tag{30}$$

From the undor field equations (19) and (21) we get the following differential equations for the adjoint undors in the symmetrical and neutral theory, respectively:

$$2\varkappa \boldsymbol{\Psi}^{\dagger} + 2\boldsymbol{X}^{\dagger} - \boldsymbol{\Psi}^{\dagger} D_{\mu} \left(\gamma_{\mu}^{(1)} - \gamma_{\mu}^{(2)}\right) = 0 \tag{31}$$

$$2\varkappa \Psi^{\dagger} + 2X^{\dagger} - \Psi^{\dagger} D_{\mu} \left(\gamma_{\mu}^{(1)} + \gamma_{\mu}^{(2)}\right) = 0. \tag{32}$$

8. The Lagrangeian and the current vector in the symmetrical theory.

The field equations (19) may be derived from the variational principle

$$\delta \int \mathscr{L}' \, d\Omega = 0 \tag{33}$$

where

$$\mathcal{L}' = \boldsymbol{\Psi}^\dagger \left\{ \left(\kappa + \frac{\overset{(1)}{\gamma_\mu} - \overset{(2)}{\gamma_\mu}}{2} D_\mu \right) \boldsymbol{\Psi} + 2\,\boldsymbol{X} \right\} \tag{34}$$

the 16 quantities $\boldsymbol{\Psi}_{k_1 k_2}$ being varied independently. In the expression (34) a scalar product like $\boldsymbol{\Psi}^\dagger \boldsymbol{\Psi}$ means

$$\boldsymbol{\Psi}^\dagger \boldsymbol{\Psi} = \sum_{k_1, k_2 = 1}^{4} \boldsymbol{\Psi}^\dagger_{k_1 k_2} \boldsymbol{\Psi}_{k_1 k_2}. \tag{35}$$

The function $\mathcal{L}'$ can, however, not be considered as a Lagrangeian. In fact, if we express $\mathcal{L}'$ in terms of the tensor field variables, we obtain

$$\begin{aligned}\mathcal{L}' = \kappa \Big\{ & -\frac{1}{2} \boldsymbol{G}_{\mu\nu} \boldsymbol{G}_{\mu\nu} + \kappa^2 \boldsymbol{U}_\mu \boldsymbol{U}_\mu + \boldsymbol{K}^2 + \boldsymbol{G}_{\mu\nu} \boldsymbol{S}_{\mu\nu} - \\ & -2 \boldsymbol{U}_\mu \boldsymbol{M}_\mu - 2 \boldsymbol{K}\boldsymbol{C} + \boldsymbol{U}_\mu D_\nu \boldsymbol{G}_{\mu\nu} + \frac{1}{2} \boldsymbol{G}_{\mu\nu} (D_\mu \boldsymbol{U}_\nu - D_\nu \boldsymbol{U}_\mu) \Big\}\end{aligned} \tag{36}$$

where now $\boldsymbol{G}_{\mu\nu}$, $\boldsymbol{U}_\mu$ and $\boldsymbol{K}$ should be varied independently, but all components of $\boldsymbol{G}_{\mu\nu}$ cannot be regarded as canonical variables, since some of the corresponding time derivatives do not appear in $\mathcal{L}'$. We have, therefore, to proceed otherwise and to regard the field equations (1 a) as definitions of the $\boldsymbol{G}_{\mu\nu}$, so that the quantities $\boldsymbol{U}_\mu$, only, are independent variables in (36). At the same time, we put the scalars $\boldsymbol{K}$ and $\boldsymbol{C}$ equal to zero, these quantities being unimportant in this connection. Substituting the expressions (1 a) for $\boldsymbol{G}_{\mu\nu}$ in (36) we obtain, by partial integration, the Lagrangeian

$$\mathcal{L} = \kappa \left(\frac{1}{2} \boldsymbol{G}_{\mu\nu} \boldsymbol{G}_{\mu\nu} + \kappa^2 \boldsymbol{U}_\mu \boldsymbol{U}_\mu - 2\, \boldsymbol{U}_\mu \boldsymbol{M}_\mu \right). \tag{37}$$

Expressing the equations (1 a) in undor language, these relations are now to be regarded as subsidiary conditions in the variational principle (33). As the total Lagrangeian we get thus

$$\left.\begin{aligned}\bar{\mathscr{L}} = \int_{\Omega} \Big\{ & \boldsymbol{\Psi}^{\dagger} \left[\left(\kappa + \frac{\gamma_{\mu}^{(1)} - \gamma_{\mu}^{(2)}}{2} D_{\mu} \right) \boldsymbol{\Psi} + 2 \boldsymbol{X} \right] + \\ & + 2 \frac{\hbar c}{i} \psi^{\dagger} \left(\gamma_{\mu} D_{\mu} + \frac{M_0 c}{\hbar} \right) \psi \Big\} d\Omega \end{aligned}\right\} \quad (38)$$

or, in terms of the tensor field variables:

$$\left.\begin{aligned}\bar{\mathscr{L}} = \int_{\Omega} \kappa \Big\{ & \frac{1}{2} \boldsymbol{G}_{\mu\nu} \boldsymbol{G}_{\mu\nu} + \kappa^2 \boldsymbol{U}_{\mu} \boldsymbol{U}_{\mu} - 2 \boldsymbol{U}_{\mu} \boldsymbol{M}_{\mu} + \\ & + 2 \frac{\hbar c}{i} \psi^{\dagger} \left(\gamma_{\mu} D_{\mu} + \frac{M_0 c}{\hbar} \right) \psi \Big\} d\Omega \end{aligned}\right\} \quad (39)$$

where M_0 is an abbreviation for

$$M_0 = \frac{1 + \tau_3}{2} M_N + \frac{1 - \tau_3}{2} M_P,$$

M_N and M_P denoting the masses of the neutron and the proton, respectively.

Then we have, in accordance with M. (21) and (22), the variational principle

$$\delta \bar{\mathscr{L}} = 0 \quad (40)$$

where ψ, $\psi^{\dagger}$ and $\boldsymbol{U}_{\mu}$ should be varied independently in such a way that the variations are zero at the boundary of the region Ω. The Euler equations corresponding to variations of the $\boldsymbol{U}_{\mu}$ are, as already mentioned, identical with the field equations (16). Similarly, variation of ψ and $\psi^{\dagger}$ leads to the wave-equations of the nucleons. In this derivation, we can utilize the undor form (38) of $\bar{\mathscr{L}}$. Remembering that the undor $\boldsymbol{X}$, only, and not $\boldsymbol{\Psi}$ depends upon ψ and $\psi^{\dagger}$, we get

$$\left\{ \gamma_{\mu} D_{\mu} + \frac{M_0 c}{\hbar} \right\} \psi + \frac{i}{\hbar c} \boldsymbol{\Psi}^{\dagger} \frac{\partial \boldsymbol{X}}{\partial \psi^{\dagger}} = 0 \quad (41)$$

$$\psi^\dagger \left\{ D_\mu \gamma_\mu - \frac{M_0 c}{\hbar} \right\} - \frac{i}{\hbar c} \boldsymbol{\Psi}^\dagger \frac{\partial \boldsymbol{X}}{\partial \psi} = 0 \tag{42}$$

with

$$\boldsymbol{\Psi}^\dagger \frac{\partial \boldsymbol{X}}{\partial \psi} = \sum_{k_1, k_2 = 1}^{4} \boldsymbol{\Psi}^\dagger_{k_1 k_2} \frac{\partial \boldsymbol{X}_{k_1 k_2}}{\partial \psi}.$$

In terms of the tensor field variables these equations become

$$\left\{ \gamma_\mu D_\mu + \frac{M_0 c}{\hbar} - \frac{i g_1}{\hbar c} \boldsymbol{U}_\mu \boldsymbol{\tau} \gamma_\mu + \frac{i g_2}{4 \kappa \hbar c} \boldsymbol{G}_{\mu\nu} \boldsymbol{\tau} \left[\gamma_\mu, \gamma_\nu \right] \right\} \psi = 0 \tag{43}$$

$$\psi^\dagger \left\{ D_\mu \gamma_\mu - \frac{M_0 c}{\hbar} + \frac{i g_1}{\hbar c} \boldsymbol{U}_\mu \boldsymbol{\tau} \gamma_\mu - \frac{i g_2}{4 \kappa \hbar c} \boldsymbol{G}_{\mu\nu} \boldsymbol{\tau} \left[\gamma_\mu, \gamma_\nu \right] \right\} = 0 \tag{44}$$

which are the same as (15) and (15′) in M.

Multiplying (41) by $\psi^\dagger$ from the left and (42) by ψ from the right, and adding, we get

$$D_\mu \psi^\dagger \gamma_\mu \psi = 0. \tag{45}$$

Furthermore, multiplying (41) by $\frac{1}{2} \psi^\dagger \tau_3$ from the left and (42) by $\frac{1}{2} \tau_3 \psi$ from the right, and adding, we obtain, by using the ordinary commutation relations of the isotopic spin matrices[1]:

$$D_\mu \frac{1}{2} \psi^\dagger \tau_3 \gamma_\mu \psi = \frac{1}{\hbar c} (\boldsymbol{\Psi}^\dagger \wedge \boldsymbol{X})_3 . \tag{46}$$

Here we have used the formulae

$$\frac{\partial \boldsymbol{X}}{\partial \psi} \psi = \boldsymbol{X}, \qquad \psi^\dagger \frac{\partial \boldsymbol{X}}{\partial \psi^\dagger} = \boldsymbol{X}$$

indicating that $\boldsymbol{X}$ is a homogeneous function of the first degree in ψ and $\psi^\dagger$, respectively.

[1] The symbol $\wedge$ indicates, as in MR. and M., a symbolical vector multiplication, e. g. $(\boldsymbol{A} \wedge \boldsymbol{B})_3 = A_1 B_2 - A_2 B_1$.

In a similar way, we get from (19) and (31), using (29) and (30),

$$\left.\begin{aligned} & D_\mu\left(\boldsymbol{\Psi}^\dagger \wedge \frac{\gamma_\mu^{(1)} - \gamma_\mu^{(2)}}{2}\boldsymbol{\Psi}\right) = \\ = & \left(\boldsymbol{\Psi}^\dagger D_\mu \frac{\gamma_\mu^{(1)} - \gamma_\mu^{(2)}}{2}\right) \wedge \boldsymbol{\Psi} + \boldsymbol{\Psi}^\dagger \wedge \left(\frac{\gamma_\mu^{(1)} - \gamma_\mu^{(2)}}{2} D_\mu \boldsymbol{\Psi}\right) = \\ & = \boldsymbol{X}^\dagger \wedge \boldsymbol{\Psi} - \boldsymbol{\Psi}^\dagger \wedge \boldsymbol{X} = -2\left(\boldsymbol{\Psi}^\dagger \wedge \boldsymbol{X}\right). \end{aligned}\right\} \quad (47)$$

From (45), (46), and (47) we finally see that the five-vector

$$S_\mu = \psi^\dagger \frac{1-\tau_3}{2}\psi - \frac{1}{4\hbar c}\left(\boldsymbol{\Psi}^\dagger \wedge \left(\gamma_\mu^{(1)} - \gamma_\mu^{(2)}\right)\boldsymbol{\Psi}\right)_3 \quad (48)$$

satisfies the divergence relation

$$D_\mu S_\mu = 0.$$

Thus, the vectors $\psi^\dagger \gamma_\mu \psi$ and S_μ may be interpreted as the five-dimensional particle-current density and charge-current density, respectively.

The expression (48) for the charge-current density is obviously the only possible. It is true that also the quantities

$$\Psi_i^\dagger \frac{\gamma_\mu^{(1)} - \gamma_\mu^{(2)}}{2} \Psi_i \quad (49)$$

have vanishing divergence but, on account of (15) and (30), it is seen that

$$\boldsymbol{\Psi}^\dagger \frac{\gamma_\mu^{(1)} - \gamma_\mu^{(2)}}{2}\boldsymbol{\Psi} = \boldsymbol{U}_\nu \boldsymbol{G}_{\nu\mu} + \frac{1}{2}\left(\boldsymbol{G}_{\mu\nu}\boldsymbol{U}_\nu - \boldsymbol{G}_{\nu\mu}\boldsymbol{U}_\nu\right) = 0 \quad (50)$$

and, consequently, all quantities (49) are identically equal to zero.

9. Difficulties of the five-dimensional neutral meson theory.

If we try to establish a variational principle of the neutral meson theory given by the equations (21), (22), and (23) in the same way as it was done in section 8 for the symmetrical theory, we meet with a serious difficulty. Since the equations (21) of the neutral meson theory differ from the equations (19) of the symmetrical theory by the sign of $\gamma_\mu^{(2)}$, only, one would expect that a variational principle of the neutral theory would be obtained from (33) by changing the sign of $\gamma_\mu^{(2)}$ in (34). Such a procedure, however, does not lead to the right field equation (23), but we get a set of trivial equations in which the terms containing the operators Grad, Div and Rot in (23) are missing.

The reason for this peculiar difference between the two theories is the following. In section 8, we have in the derivation of the field equations from the variational principle made use of the relation

$$D_\mu \left\{ \mathbf{\Psi}^\dagger \frac{\partial \dfrac{\gamma_\nu^{(1)} - \gamma_\nu^{(2)}}{2} D_\nu \mathbf{\Psi}}{\partial \left(D_\mu \mathbf{\Psi}_{k_1 k_2} \right)} \right\} = -\left(\frac{\gamma_\mu^{(1)} - \gamma_\mu^{(2)}}{2} D_\mu \mathbf{\Psi} \right)_{k'_1 k'_2} \tag{51}$$

where k_1', k_2' are determined by (cf. (30))

$$\mathbf{\Psi}^\dagger_{k'_1 k'_2} = \pm \mathbf{\Psi}_{k_1 k_2}.$$

The analogous relation in the neutral theory is

$$D_\mu \left\{ \Psi^\dagger \frac{\partial \dfrac{\gamma_\nu^{(1)} + \gamma_\nu^{(2)}}{2} D_\nu \Psi}{\partial \left(D_\mu \Psi_{k_1 k_2} \right)} \right\} = \left(\frac{\gamma_\mu^{(1)} + \gamma_\mu^{(2)}}{2} D_\mu \Psi \right)_{k'_1 k'_1} \tag{52}$$

and the difference in sign of the right-hand sides in (51) and (52) makes in the Euler equations all terms containing derivatives cancel each other.

The four-dimensional theory described by (25) may, however, be derived from a variational principle with the Lagrangeian

$$\mathcal{L} = \frac{\kappa}{3!}\left(\frac{1}{2} G_{klm} G_{klm} + \frac{3}{2} G_{kl} G_{kl} - G_{klm} S_{klm} - \frac{\kappa^2}{2} U_k U_k - \frac{1}{2} K^2 + KC\right) \quad (53)$$

K and G_{klm} being the independent variables, while U_k and G_{kl} are defined by (25 b) and (25 e), respectively.

Since

$$G_{\mu\nu} G_{\mu\nu} = G_{kl} G_{kl} + \frac{1}{3} G_{klm} G_{klm} \quad (54)$$

the direct generalization of the Lagrangeian to five-dimensions will be

$$\mathcal{L} = \frac{\kappa}{2}\left\{\frac{1}{2} G_{\mu\nu} G_{\mu\nu} - G_{\mu\nu} S_{\mu\nu} - \kappa^2 U_\mu U_\mu - K^2 + 2 KC\right\}. \quad (55)$$

If we vary K, the U_μ being defined by means of (23 b), we really obtain the equation (23 a); but if we vary all the $G_{\mu\nu}$ independently—which would be a possible generalization of the four-dimensional case—we get the equations

$$G_{\mu\nu} = S_{\mu\nu} \quad (56)$$

which differ from (23 c) by the essential term Rot G.

On the other hand, if we also in the five-dimensional theory consider the quantities G_{klm} as independent variables, the G_{kl} being defined by (25 e), we obtain the equations

$$\left.\begin{aligned} &\kappa\left(G_{klm} - S_{klm}\right) = \\ &= D_k G_{lm} + D_l G_{mk} + D_m G_{kl} - D_k S_{lm} - D_l S_{mk} - D_m S_{kl}, \end{aligned}\right\} \quad (57)$$

which deviate from the field equations (25 d) by terms containing derivatives of the source densities.

It is easily seen that it is impossible to find any kind of variational principle of the type (33) from which the right field equations could be derived by varying all the quantities $G_{\mu\nu}$ independently. Since the corresponding function $\mathcal{L}'$ has to be an invariant, it will, in the general case, be of the shape

$$\mathcal{L}' = a\, G_{\mu\nu}\, G_{\mu\nu} + b\, G_{\mu\nu}\, \mathrm{S}_{\mu\nu} + c\, \mathrm{Rot}_{\mu\nu}\, G \cdot G_{\mu\nu}$$

where a, b and c are constants. If $c = 0$, we obviously have to do with the case treated above, but even if $c \neq 0$, the last term will give no contribution to the EULER equation since the variation of

$$\mathrm{Rot}_{\mu\nu} G \cdot G_{\mu\nu}$$

on account of the definition (24) is equal to zero for any variation of the quantities $G_{\mu\nu}$ which vanish at the boundary.

On the other hand, if we only vary the G_{klm}, the G_{kl} being again defined by (25 c), the constant c has also to be put equal to zero; otherwise we would get field equations involving derivatives of the second order, since we have (cf. (54))

$$\left.\begin{aligned} G_{\mu\nu}\, \mathrm{Rot}_{\mu\nu}\, G &= G_{kl}\, \mathrm{Rot}_{kl}\, G + \frac{1}{3}\, G_{klm}\, \mathrm{\rho o \tau}_{klm}\, G = \\ &= \frac{1}{\kappa}\, D_r\, G_{klr}\, \mathrm{Rot}_{kl}\, G + S_{kl}\, \mathrm{Rot}_{kl}\, G + \\ &+ \frac{1}{3}\, G_{klm}\left(D_k\, G_{lm} + D_m\, G_{kl} + D_l\, G_{mk}\right) \end{aligned}\right\} \quad (58)$$

where the quantity $\mathrm{Rot}_{kl} G$ contains terms like $D_m\, G_{klm}$ (no summation!). Consequently, the first term on the right hand of (58) will give rise to second derivatives in the Euler equation.

Thus, the equations of the five-dimensional neutral meson theory cannot be derived from any variational principle, a circumstance which may be connected with the fact that the field equations (21)-(23) consist of two separate groups: the scalar equations (a), (b) and the tensor equations (c) which even under five-dimensional rotations are not transformed into each other. Since a Lagrangeian principle is essential for the quantization of the field equations, the impossibility of formulating such principle in the neutral theory may be taken as a strong argument in favour of the symmetrical theory which, in contrast to the neutral theory, is able to satisfy the more rigorous claims resulting from a generalization to five dimensions.

I wish to express my deep gratitude to Professor Niels Bohr for the kind interest shown in this investigation and for the possibilities given to me to work in the Institute of Theoretical Physics, Copenhagen. Furthermore, my most hearty thanks are due Dr. C. Møller for his constant guidance and for many valuable discussions during the performance of the present paper.

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Indleveret til Selskabet den 13. September 1941.
Færdig fra Trykkeriet den 12. Januar 1942.